

# Symmetric Nonnegative Tensors and Copositive Tensors

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## Abstract

We first prove two new spectral properties for symmetric nonnegative tensors. We prove a maximum property for the largest H-eigenvalue of a symmetric nonnegative tensor, and establish some bounds for this eigenvalue via row sums of that tensor. We show that if an eigenvalue of a symmetric nonnegative tensor has a positive H-eigenvector, then this eigenvalue is the largest H-eigenvalue of that tensor. We also give a necessary and sufficient condition for this. We then introduce copositive tensors. This concept extends the concept of copositive matrices. Symmetric nonnegative tensors and positive semi-definite tensors are examples of copositive tensors. The diagonal elements of a copositive tensor must be nonnegative. We show that if each sum of a diagonal element and all the negative off-diagonal elements in the same row of a real symmetric tensor is nonnegative, then that tensor is a copositive tensor. Some further properties of copositive tensors are discussed.

**Key words:** nonnegative tensor, copositive tensor, H-eigenvalue

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## 1 Introduction

Eigenvalues of higher-order tensors were introduced in [14, 10] in 2005. Since then, many research works have been done in spectral theory of tensors. In particular, the theory and algorithms for eigenvalues of nonnegative tensors are well developed [2, 3, 5, 9, 11, 13, 18, 19, 20, 22].

In this paper, we prove two new spectral properties for symmetric nonnegative tensors. Then we introduce copositive tensors, establish some necessary conditions and some sufficient

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conditions for a real symmetric tensor to be a copositive tensor, and discuss some further properties of copositive tensors.

Some preliminary concepts and results are given in the next section.

We prove a maximum property of the largest H-eigenvalue of a symmetric nonnegative tensor in Section 3. Based upon this, we establish some bounds for the largest eigenvalue of a symmetric nonnegative tensor via row sums of that tensor.

In Section 4, we show that a symmetric nonnegative tensor has at most one  $H^{++}$ -eigenvalue, i.e., an H-eigenvalue with a positive H-eigenvector, and if an eigenvalue of a symmetric nonnegative tensor has a positive H-eigenvector, then that eigenvalue must equal to the largest eigenvalue of that tensor. We also give a necessary and sufficient condition for the existence of such an  $H^{++}$ -eigenvalue.

In Section 5, we introduce copositive tensors and strictly copositive tensors. These two concepts extend the concepts of copositive matrices and strictly copositive matrices. Symmetric nonnegative tensors and positive semi-definite tensors are examples of copositive tensors. The diagonal elements of a copositive tensor must be nonnegative. We show that if each sum of a diagonal element and all the negative off-diagonal elements in the same row of a real symmetric tensor is nonnegative, then that tensor is a copositive tensor.

Some further properties of copositive tensors are discussed in Section 6. We show that if a copositive tensor has an  $H^+$ -eigenvalue, i.e., an H-eigenvalue with a nonnegative H-eigenvector, then that  $H^+$ -eigenvalue must be nonnegative. The sets of copositive tensors and strictly copositive tensors form two convex cones: the copositive tensor cone and the strictly copositive tensor cone. We show that the latter is exactly the interior of the former. We also introduced completely positive tensors. The copositive tensor cone is the dual cone of the completely positive tensor cone. If the completely positive tensor cone is closed, then these two cones are dual to each other.

Some final remarks are made in Section 7.

Denote by  $e$  the all 1  $n$ -dimensional vector,  $e_j = 1$  for  $j = 1, \dots, n$ . Denote by  $e^{(i)}$  the  $i$ th unit vector in  $\mathbb{R}^n$ , i.e.,  $e_j^{(i)} = 1$  if  $i = j$  and  $e_j^{(i)} = 0$  if  $i \neq j$ , for  $i, j = 1, \dots, n$ . Denote the set of all nonnegative vectors in  $\mathbb{R}^n$  by  $\mathbb{R}_+^n$  and the set of all positive vectors in  $\mathbb{R}^n$  by  $\mathbb{R}_{++}^n$ . If both  $\mathcal{A} = (a_{i_1 \dots i_k})$  and  $\mathcal{B} = (b_{i_1 \dots i_k})$  are real  $k$ th order  $n$ -dimensional tensors, and  $b_{i_1 \dots i_k} \leq a_{i_1 \dots i_k}$  for  $i_1, \dots, i_k = 1, \dots, n$ , then we denote  $\mathcal{B} \leq \mathcal{A}$ . We use  $\mathcal{J}$  to denote the  $k$ th order  $n$ -dimensional tensor with all of its elements being 1. We use  $\mathcal{I}$  to denote the  $k$ th order  $n$ -dimensional diagonal tensor with all of its diagonal elements being 1.

## 2 Preliminaries

Let  $\mathcal{A} = (a_{i_1 \dots i_k})$  be a real  $k$ th order  $n$ -dimensional tensor, and  $x \in C^n$ . Then

$$\mathcal{A}x^k = \sum_{i_1, \dots, i_k=1}^n a_{i_1 \dots i_k} x_{i_1} \cdots x_{i_k},$$

and  $\mathcal{A}x^{k-1}$  is a vector in  $C^n$ , with its  $i$ th component defined by

$$(\mathcal{A}x^{k-1})_i = \sum_{i_2, \dots, i_k=1}^n a_{i i_2 \dots i_k} x_{i_2} \cdots x_{i_k}.$$

Let  $r$  be a positive integer. Then  $x^{[r]}$  is a vector in  $C^n$ , with its  $i$ th component defined by  $x_i^r$ . We say that  $\mathcal{A}$  is symmetric if its entries  $a_{i_1, \dots, i_k}$  are invariant for any permutation of the indices.

If  $x \in C^n$ ,  $x \neq 0$ ,  $\lambda \in C$ ,  $x$  and  $\lambda$  satisfy

$$\mathcal{A}x^{k-1} = \lambda x^{[k-1]}, \quad (1)$$

then we call  $\lambda$  an **eigenvalue** of  $\mathcal{A}$ , and  $x$  its corresponding **eigenvector**. By (1), if  $\lambda$  is an eigenvalue of  $\mathcal{A}$  and  $x$  is its corresponding eigenvector, then

$$\lambda = \frac{(\mathcal{A}x^{k-1})_j}{x_j^{k-1}},$$

for some  $j$  with  $x_j \neq 0$ . In particular, if  $x$  is real, then  $\lambda$  is also real. In this case, we say that  $\lambda$  is an **H-eigenvalue** of  $\mathcal{A}$  and  $x$  is its corresponding **H-eigenvector**. If  $x \in \mathfrak{R}_+^n$ , then we say that  $\lambda$  is an **H<sup>+</sup>-eigenvalue** of  $\mathcal{A}$ . If  $x \in \mathfrak{R}_{++}^n$ , then we say that  $\lambda$  is an **H<sup>++</sup>-eigenvalue** of  $\mathcal{A}$ . The largest modulus of the eigenvalues of  $\mathcal{A}$  is called the **spectral radius** of  $\mathcal{A}$ , denoted by  $\rho(\mathcal{A})$ .

By [2],  $\mathcal{A}$  is called **reducible** if there exists a proper nonempty subset  $I$  of  $\{1, \dots, n\}$  such that

$$a_{i_1 \dots i_k} = 0, \quad \forall i_1 \in I, \quad \forall i_2, \dots, i_k \notin I.$$

If  $\mathcal{A}$  is not reducible, then we say that  $\mathcal{A}$  is **irreducible**.

Let  $\mathcal{A} = (a_{i_1, \dots, i_k})$  be a  $k$ th order  $n$ -dimensional symmetric nonnegative tensor. Construct a graph  $\hat{G}(\mathcal{A}) = (\hat{V}, \hat{E})$ , where  $\hat{V} = \cup_{j=1}^n V_j$ ,  $V_j$  is a copy of  $\{1, \dots, n\}$ , for  $j = 1, \dots, n$ . Assume that  $i_j \in V_j, i_l \in V_l, j \neq l$ . The edge  $(i_j, i_l) \in \hat{E}$  if and only if  $a_{i_1, \dots, i_k} \neq 0$  for some  $k-2$  indices  $\{i_1, \dots, i_k\} \setminus \{i_j, i_l\}$ . The tensor  $\mathcal{A}$  is called **weakly irreducible** if  $\hat{G}(\mathcal{A})$  is connected. As observed in [5], an irreducible symmetric nonnegative tensor is always weakly irreducible.

Suppose that  $\mathcal{A} = (a_{i_1, \dots, i_k})$  is a  $k$ th order  $n$ -dimensional real tensor. If all the off-diagonal entries  $a_{i_1 \dots i_k}, (i_1, \dots, i_k) \neq (i_1, \dots, i_1)$  are nonnegative, then  $\mathcal{A}$  is called an **essentially non-negative tensor** [21]. If all the entries  $a_{i_1 \dots i_k}$  are nonnegative,  $\mathcal{A}$  is called a **nonnegative tensor**. We now summarize the Perron-Frobenius theorem for nonnegative tensors, established in [2, 5, 18]. With the new definitions of  $H^+$ -eigenvalues and  $H^{++}$ -eigenvalues, this theorem can be stated concisely.

**Theorem 1 (The Perron-Frobenius Theorem for Nonnegative Tensors)**

- (a). (Yang and Yang 2010) *If  $\mathcal{A}$  is a nonnegative tensor of order  $k$  and dimension  $n$ , then  $\rho(\mathcal{A})$  is an  $H^+$ -eigenvalue of  $\mathcal{A}$ .*
- (b). (Friedland, Gaubert and Han 2011) *If furthermore  $\mathcal{A}$  is symmetric and weakly irreducible, then  $\rho(\mathcal{A})$  is the unique  $H^{++}$ -eigenvalue of  $\mathcal{A}$ , with the unique eigenvector  $x \in \mathbb{R}_{++}^n$ , up to a positive scaling constant.*
- (c). (Chang, Pearson and Zhang 2008) *If moreover  $\mathcal{A}$  is irreducible, then  $\rho(\mathcal{A})$  is the unique  $H^+$ -eigenvalue of  $\mathcal{A}$ .*
- (d). (Yang and Yang 2010) *If  $\mathcal{A}$  and  $\mathcal{B}$  are two nonnegative tensor of order  $k$  and dimension  $n$ , and  $\mathcal{B} \leq \mathcal{A}$ , then  $\rho(\mathcal{B}) \leq \rho(\mathcal{A})$ .*

Thus, for a nonnegative tensor  $\mathcal{A}$ , its spectral radius is its largest  $H$ -eigenvalue.

As observed in [21], a tensor  $\mathcal{A}$  is an essentially nonnegative tensor, if and only if there are a nonnegative tensor  $\mathcal{B}$  and a real number  $c$ , such that  $\mathcal{A} = \mathcal{B} + c\mathcal{I}$ .

### 3 A Maximum Property of the Largest $H$ -Eigenvalue of a Symmetric Nonnegative Tensor

Suppose that  $\mathcal{A} = (a_{i_1 \dots i_k})$  is a  $k$ th order  $n$ -dimensional real symmetric tensor, with  $k \geq 2$ . Denote its largest  $H$ -eigenvalue by  $\lambda_{\max}(\mathcal{A})$ . When  $k$  is even, by [14], we know that

$$\lambda_{\max}(\mathcal{A}) = \max\{\mathcal{A}x^k : x \in \mathbb{R}^n, \sum_{i=1}^n x_i^k = 1\}.$$

In this section, we first prove the following theorem, which holds whenever  $k$  is even or odd.

**Theorem 2 (A Maximum Property of The Largest  $H$ -Eigenvalue of a Symmetric Nonnegative Tensor)** *Suppose that  $\mathcal{A} = (a_{i_1 \dots i_k})$  is a  $k$ th order  $n$ -dimensional symmetric nonnegative tensor, with  $k \geq 2$ . Then we have*

$$\lambda_{\max}(\mathcal{A}) = \max\{\mathcal{A}x^k : x \in \mathbb{R}_+^n, \sum_{i=1}^n x_i^k = 1\}. \quad (2)$$

**Proof.** We now prove (2). Assume that  $\mathcal{A}$  is a symmetric nonnegative tensor. By Theorem 1, we have

$$\begin{aligned}\lambda_{\max}(\mathcal{A}) &= \max\{\lambda : \mathcal{A}x^{k-1} = \lambda x^{[k-1]}, x \in \mathfrak{R}_+^n\} = \max\{\lambda : \mathcal{A}x^{k-1} = \lambda x^{[k-1]}, x \in \mathfrak{R}_+^n, \sum_{i=1}^n x_i^k = 1\} \\ &= \max\{\mathcal{A}x^k : \mathcal{A}x^{k-1} = \lambda x^{[k-1]}, x \in \mathfrak{R}_+^n, \sum_{i=1}^n x_i^k = 1\} \leq \max\{\mathcal{A}x^k : x \in \mathfrak{R}_+^n, \sum_{i=1}^n x_i^k = 1\}.\end{aligned}$$

On the other hand, assume that  $x^*$  is an optimal solution of  $\max\{\mathcal{A}x^k : x \in \mathfrak{R}_+^n, \sum_{i=1}^n x_i^k = 1\}$ . By optimization theory, there is a Lagrangian multiplier  $\lambda$  and a nonempty subset  $I$  of  $\{1, \dots, n\}$  such that for  $i \in \{1, \dots, n\} \setminus I$ ,  $x_i^* = 0$ , and for  $i \in I$ ,

$$(\mathcal{A}(x_i^*)^{k-1})_i = \lambda (x_i^*)^{k-1}.$$

Multiplying the above equalities by  $x_i^*$  and summing up them, we have

$$\lambda = \mathcal{A}(x^*)^k = \max\{\mathcal{A}x^k : x \in \mathfrak{R}_+^n, \sum_{i=1}^n x_i^k = 1\}.$$

Construct a  $k$ th order  $n$ -dimensional symmetric nonnegative tensor  $\mathcal{B} = (b_{i_1 \dots i_k})$  such that  $b_{i_1 \dots i_k} = a_{i_1 \dots i_k}$  if  $i_1, \dots, i_k \in I$ , and  $b_{i_1 \dots i_k} = 0$  otherwise. Then we see that  $\lambda$  is an H-eigenvalue of  $\mathcal{B}$  with an H-eigenvector  $x^*$ . Then we see that  $\mathcal{B} \leq \mathcal{A}$ . By Theorem 1 (d), we have

$$\lambda \leq \rho(\mathcal{B}) \leq \rho(\mathcal{A}) = \lambda_{\max}(\mathcal{A}).$$

Combining these together, we have (2).  $\square$

The adjacency tensor of a uniform hypergraph is a nonnegative tensor [4]. The signless Laplacian tensor of a uniform hypergraph, introduced in [15], is also a nonnegative tensor. Cooper and Dutle [4] established (2) for the adjacency tensor of a connected uniform hypergraph. Qi [15] established (2) for the adjacency tensor and the signless Laplacian tensor of a general uniform hypergraph. Zhang [23] pointed out that (2) holds for a weakly irreducible symmetric nonnegative tensor. Here, we established (2) for a general symmetric nonnegative tensor.

With Theorem 2, we may establish some lower bounds for  $\rho(\mathcal{A})$ . We define the  $i$ th **row sum** of a  $k$ th order  $n$ -dimensional tensor  $\mathcal{A} = (a_{i_1, \dots, i_k})$  as

$$R_i(\mathcal{A}) = \sum_{i_2, \dots, i_k=1}^n a_{ii_2 \dots i_k},$$

and denote the largest, the smallest and the average row sums of  $\mathcal{A}$  by

$$R_{\max}(\mathcal{A}) = \max_{i=1, \dots, n} R_i(\mathcal{A}), \quad R_{\min}(\mathcal{A}) = \min_{i=1, \dots, n} R_i(\mathcal{A}), \quad \text{and} \quad \bar{R}(\mathcal{A}) = \frac{1}{n} \sum_{i=1}^n R_i(\mathcal{A}),$$

respectively. We also denote the largest, the smallest and the average diagonal element of  $\mathcal{A}$  by

$$d_{\max}(\mathcal{A}) = \max_{i=1,\dots,n} a_{i\dots i}, \quad d_{\min}(\mathcal{A}) = \min_{i=1,\dots,n} a_{i\dots i}, \quad \bar{d}(\mathcal{A}) = \frac{1}{n} \sum_{i=1}^n a_{i\dots i},$$

respectively.

**Theorem 3 (Bounds The Largest H-Eigenvalue of a Nonnegative Tensor)** *Suppose that  $\mathcal{A} = (a_{i_1\dots i_k})$  is a  $k$ th order  $n$ -dimensional nonnegative tensor, with  $k \geq 2$ . Then we have*

$$\lambda_{\max}(\mathcal{A}) \leq R_{\max}(\mathcal{A}). \quad (3)$$

*If furthermore  $\mathcal{A}$  is symmetric, then we have*

$$\lambda_{\max}(\mathcal{A}) \geq \max\{\bar{R}(\mathcal{A}), d_{\max}(\mathcal{A})\}. \quad (4)$$

**Proof.** By Theorem 1,  $\mathcal{A}$  has a nonnegative H-eigenvector  $x$ . Let  $x_j = \max_{i=1,\dots,n} x_i$ . Then  $x_j > 0$ . We have

$$\sum_{i_2,\dots,i_k=1}^n a_{ji_2\dots i_k} x_{i_2} \cdots x_{i_k} = \lambda_{\max}(\mathcal{A}) x_j^{k-1},$$

i.e.,

$$\lambda_{\max}(\mathcal{A}) = \sum_{i_2,\dots,i_k=1}^n a_{ji_2\dots i_k} \frac{x_{i_2}}{x_j} \cdots \frac{x_{i_k}}{x_j} \leq R_j(\mathcal{A}) \leq R_{\max}(\mathcal{A}).$$

This proves (3).

Now assume that  $\mathcal{A}$  is symmetric. Let  $y = \frac{e}{(n)^{\frac{1}{k}}}$ . By Theorem 2, we have

$$\lambda_{\max}(\mathcal{A}) \geq \mathcal{A}y^k = \frac{1}{n} \sum_{i_1,\dots,i_k=1}^n a_{i_1\dots i_k} = \bar{R}(\mathcal{A}).$$

Assume that  $a_{j\dots j} = d_{\max}(\mathcal{A})$ . Let  $y = e^{(j)}$ . By Theorem 2, we have

$$\lambda_{\max}(\mathcal{A}) \geq \mathcal{A}y^k = a_{j\dots j} = d_{\max}(\mathcal{A}).$$

Combining these two inequalities, we have (4).  $\square$

If we apply Theorem 3 to the adjacency tensor of a uniform hypergraph, we may get the bounds for the largest H-eigenvalue of that tensor, obtained by Cooper and Dutle in [4]. If we apply Theorem 3 to the signless Laplacian tensor of a uniform hypergraph, we may get the bounds for the largest H-eigenvalue of that tensor, obtained by Qi in [15].

For a  $k$ th order  $n$ -dimensional symmetric nonnegative tensor  $\mathcal{A}$ , if all of its row sums are the same, then we have  $\lambda_{\max}(\mathcal{A}) = \bar{R}(\mathcal{A})$ . The adjacency tensor and the signless Laplacian tensor of a regular  $k$ -graph are such examples [4, 15].

## 4 The $H^{++}$ -Eigenvalue of a Symmetric Nonnegative Tensor

In this section, we show that a symmetric nonnegative tensor has at most one  $H^{++}$ -eigenvalue.

Suppose that  $I \subset \{1, \dots, n\}$ . Let  $x$  be an  $n$ -dimensional vector. Then  $x(I)$  is an  $|I|$ -dimensional vector with its components indexed for  $i \in I$ , and  $x(I)_i \equiv x_i$  for  $i \in I$ . For a  $k$ th order  $n$ -dimensional tensor  $\mathcal{A} = (a_{i_1 \dots i_k})$ ,  $\mathcal{A}(I)$  is a  $k$ th order  $|I|$ -dimensional tensor with elements  $a_{i_1 \dots i_k}, i_1, \dots, i_k \in I$ .

Suppose that  $\mathcal{A} = (a_{i_1 \dots i_k})$  is a symmetric nonnegative tensor of order  $k$  and dimension  $n$ . By [5, 9], there is a partition  $(I_1, \dots, I_s)$  of  $\{1, \dots, n\}$ , such that  $\mathcal{A}(I_r)$  is weakly irreducible for  $r = 1, \dots, s$ , and  $a_{i_1 \dots i_k} = 0$  for all  $i_1 \in I_r, i_2, \dots, i_k \notin I_r, r = 1, \dots, s$ . Furthermore, we have

$$\lambda_{\max}(\mathcal{A}) = \max\{\lambda_{\max}(\mathcal{A}(I_r)) : r = 1, \dots, s\}.$$

**Theorem 4 (The  $H^{++}$ -Eigenvalue of a Symmetric Nonnegative Tensor)** *Let  $\mathcal{A} = (a_{i_1 \dots i_k})$  be a symmetric nonnegative tensor of order  $k$  and dimension  $n$ . Then  $\mathcal{A}$  has at most one  $H^{++}$ -eigenvalue. A real number  $\lambda$  is an  $H^{++}$ -eigenvalue of  $\mathcal{A}$  if and only if for the above partition  $(I_1, \dots, I_s)$ , we have*

$$\lambda = \lambda_{\max}(\mathcal{A}) = \lambda_{\max}(\mathcal{A}(I_r)), \text{ for } r = 1, \dots, s. \quad (5)$$

**Proof.** Suppose that (5) holds. Then by Theorem 1 (a), we have  $x \in \mathfrak{R}_+^n$  such that

$$\sum_{i_2, \dots, i_k \in I_r} a_{ii_2 \dots i_k} x_{i_2} \cdots x_{i_k} = \lambda x_i^{k-1}, \quad (6)$$

for  $i \in I_r, r = 1, \dots, s$ . By Theorem 1 (b),  $x(I_r) > 0$  for  $r = 1, \dots, s$ . Thus,  $x \in \mathfrak{R}_+^n$ . (6) further implies that

$$\sum_{i_2, \dots, i_k=1}^n a_{ii_2 \dots i_k} x_{i_2} \cdots x_{i_k} = \lambda x_i^{k-1}, \quad (7)$$

for  $i = 1, \dots, n$ , i.e.,  $\lambda$  is an  $H^{++}$ -eigenvalue of  $\mathcal{A}$ .

On the other hand, assume that  $\lambda$  is an  $H^{++}$ -eigenvalue of  $\mathcal{A}$ , with an  $H$ -eigenvector  $x \in \mathfrak{R}_{++}^n$ . Then we have (7), which implies (6). By Theorem 1 (b), we have  $\lambda = \lambda_{\max}(\mathcal{A}(I_r))$  for  $r = 1, \dots, s$ . Since  $\lambda_{\max}(\mathcal{A}) = \max\{\lambda_{\max}(\mathcal{A}(I_r)) : r = 1, \dots, s\}$ , we have (5).  $\square$

## 5 Copositive Tensors

The concept of copositive matrices was introduced by Motzkin [12] in 1952. It is an important concept in applied mathematics, with applications in control theory, optimization modeling,

linear complementarity problems, graph theory and linear evolution variational inequalities [8]. We now extend this concept to tensors.

Suppose that  $\mathcal{A} = (a_{i_1 \dots i_k})$  is a real symmetric tensor of order  $k$  and dimension  $n$ . We say that  $\mathcal{A}$  is a **copositive tensor** if for any  $x \in \mathbb{R}_+^n$ , we have  $\mathcal{A}x^k \geq 0$ . We say that  $\mathcal{A}$  is a **strictly copositive tensor** if for any  $x \in \mathbb{R}_+^n, x \neq 0$ , we have  $\mathcal{A}x^k > 0$ . Clearly, a symmetric nonnegative tensor is a copositive tensor. Recall [14] that a real symmetric tensor  $\mathcal{A}$  of order  $k$  and dimension  $n$ , is called a positive semi-definite tensor, if for any  $x \in \mathbb{R}^n$ ,  $\mathcal{A}x^k \geq 0$ ,  $\mathcal{A}$  is called a positive definite tensor, if for any  $x \in \mathbb{R}^n, x \neq 0$ ,  $\mathcal{A}x^k > 0$ . Except the zero tensor, positive semi-definite tensors are of even order. Clearly, a positive semi-definite tensor is a copositive tensor, a positive definite tensor is a strictly copositive tensor.

**Theorem 5 (Copositive Tensors)** *Suppose that  $\mathcal{A} = (a_{i_1 \dots i_k})$  and  $\mathcal{B} = (b_{i_1 \dots i_k})$  are two real symmetric tensors of order  $k$  and dimension  $n$ . Then we have the following conclusions.*

(a).  $\mathcal{A}$  is copositive if and only if

$$N_{\min}(\mathcal{A}) \equiv \min\{\mathcal{A}x^k : x \in \mathbb{R}_+^n, \sum_{i=1}^n x_i^k = 1\} \geq 0. \quad (8)$$

$\mathcal{A}$  is strictly copositive if and only if

$$N_{\min}(\mathcal{A}) \equiv \min\{\mathcal{A}x^k : x \in \mathbb{R}_+^n, \sum_{i=1}^n x_i^k = 1\} > 0. \quad (9)$$

(b). If  $\mathcal{A}$  is copositive, then  $d_{\min}(\mathcal{A}) \geq 0$ . If  $\mathcal{A}$  is strictly copositive, then  $d_{\min}(\mathcal{A}) > 0$ .

(c). Suppose that  $\mathcal{A} \leq \mathcal{B}$ . If  $\mathcal{A}$  is copositive, then  $\mathcal{B}$  is copositive. If  $\mathcal{A}$  is strictly copositive, then  $\mathcal{B}$  is strictly copositive.

**Proof.** (a). If  $\mathcal{A}$  is copositive, then clearly (8) holds. Suppose (8) holds. For any  $y \in \mathbb{R}_+^n, y \neq 0$ , let

$$x = \frac{y}{\left(\sum_{i=1}^n y_i^k\right)^{\frac{1}{k}}}.$$

Then  $x \in \mathbb{R}_+^n$ ,  $\sum_{i=1}^n x_i^k = 1$ , and

$$\mathcal{A}x^k = \frac{\mathcal{A}y^k}{\sum_{i=1}^n y_i^k} \geq \frac{N_{\min}(\mathcal{A})}{\sum_{i=1}^n y_i^k} \geq 0.$$

Thus,  $\mathcal{A}$  is copositive.

Similarly, if (9) holds, we may show that  $\mathcal{A}$  is strictly copositive. Suppose that (9) does not hold. As the feasible set of the minimization problem in (9) is compact, the minimization problem has an optimizer  $x^*$ . Then  $x^* \in \mathbb{R}_+^n, x^* \neq 0$  and  $\mathcal{A}(x^*)^k = N_{\min}(\mathcal{A}) \leq 0$ . Thus  $\mathcal{A}$  cannot be strictly copositive. This completes the proof of (a).



(b). Assume that  $d_j(\mathcal{A}) = d_{\min}(\mathcal{A})$ . Let  $y = e^{(j)}$ . Then  $y \in \mathfrak{R}_+^n$ ,  $\sum_{i=1}^n y_i^k = 1$ , and  $d_{\min}(\mathcal{A}) = \mathcal{A}y^k$ . If  $\mathcal{A}$  is copositive, then by (a),

$$d_{\min}(\mathcal{A}) = \mathcal{A}y^k \geq N_{\min}(\mathcal{A}) \geq 0.$$

If  $\mathcal{A}$  is strictly copositive, then by (a),

$$d_{\min}(\mathcal{A}) = \mathcal{A}y^k \geq N_{\min}(\mathcal{A}) > 0.$$

These prove (b).

(c). Suppose that  $\mathcal{A} \leq \mathcal{B}$ . If  $\mathcal{A}$  is copositive, then for any  $x \in \mathfrak{R}_+^n$ ,  $\mathcal{B}x^k \geq \mathcal{A}x^k \geq 0$ . This implies that  $\mathcal{B}$  is copositive. If  $\mathcal{A}$  is strictly copositive, then for any  $x \in \mathfrak{R}_+^n$ ,  $x \neq 0$ ,  $\mathcal{B}x^k \geq \mathcal{A}x^k > 0$ . This implies that  $\mathcal{B}$  is strictly copositive.  $\square$

We now prove further a nontrivial sufficient condition for a real symmetric tensor to be copositive. We need to prove some lemmas first.

**Lemma 6** *Suppose that  $\mathcal{A} = (a_{i_1 \dots i_k})$  is a  $k$ th order  $n$ -dimensional symmetric, essentially nonnegative tensor, with  $k \geq 2$ . Then we still have (2).*

**Proof.** Assume that  $\mathcal{A}$  is a symmetric, essentially nonnegative tensor. Then there are a symmetric nonnegative tensor  $\mathcal{B}$  and a real number  $c$ , such that  $\mathcal{A} = \mathcal{B} + c\mathcal{I}$ . Then,

$$\begin{aligned} \lambda_{\max}(\mathcal{A}) &= \lambda_{\max}(\mathcal{B}) + c = \max\{\mathcal{B}x^k : x \in \mathfrak{R}_+^n, \sum_{i=1}^n x_i^k = 1\} + c \\ &= \max\{\mathcal{B}x^k + c : x \in \mathfrak{R}_+^n, \sum_{i=1}^n x_i^k = 1\} = \max\{\mathcal{A}x^k : x \in \mathfrak{R}_+^n, \sum_{i=1}^n x_i^k = 1\}. \end{aligned}$$

This completes the proof.  $\square$

We may also show that Theorem 3 also holds for symmetric, essentially nonnegative tensors, and Theorem 4 also holds for essentially nonnegative tensors. We do not go to the details.

Suppose that  $\mathcal{A} = (a_{i_1, \dots, i_k})$  is a  $k$ th order  $n$ -dimensional real tensor. If all the off-diagonal entries  $a_{i_1 \dots i_k}, (i_1, \dots, i_k) \neq (i_1, \dots, i_1)$  are nonpositive, then  $\mathcal{A}$  is called an **essentially nonpositive tensor**.

For a  $k$ th order  $n$ -dimensional real tensor  $\mathcal{A}$ , denote its smallest H-eigenvalues by  $\lambda_{\min}(\mathcal{A})$ .

**Lemma 7** *Suppose that  $\mathcal{A} = (a_{i_1 \dots i_k})$  is a  $k$ th order  $n$ -dimensional symmetric, essentially nonpositive tensor, with  $k \geq 2$ . Then we have*

$$\lambda_{\min}(\mathcal{A}) = \min\{\mathcal{A}x^k : x \in \mathfrak{R}_+^n, \sum_{i=1}^n x_i^k = 1\}. \quad (10)$$

**Proof.** Assume that  $\mathcal{A}$  is a symmetric, essentially nonpositive tensor. Let  $\mathcal{B} = -\mathcal{A}$ . Then  $\mathcal{B}$  is a symmetric, essentially nonnegative tensor. Then,

$$\lambda_{\min}(\mathcal{A}) = -\lambda_{\max}(\mathcal{B}) = -\max\{\mathcal{B}x^k : x \in \mathbb{R}_+^n, \sum_{i=1}^n x_i^k = 1\} = \min\{\mathcal{A}x^k : x \in \mathbb{R}_+^n, \sum_{i=1}^n x_i^k = 1\}.$$

This completes the proof.  $\square$

**Lemma 8** Suppose that  $\mathcal{A} = (a_{i_1 \dots i_k})$  is a  $k$ th order  $n$ -dimensional essentially nonpositive tensor, with  $k \geq 2$ . Then we have

$$\lambda_{\min}(\mathcal{A}) \geq R_{\min}(\mathcal{A}).$$

If furthermore  $\mathcal{A}$  is symmetric, then we have

$$\lambda_{\min}(\mathcal{A}) \leq \min\{\bar{R}(\mathcal{A}), d_{\min}(\mathcal{A})\}.$$

The proof of this lemma is similar to the proof of Theorem 3. By Lemma 8 and Theorem 5 (a), we have the following lemma.

**Lemma 9** Suppose that  $\mathcal{A} = (a_{i_1 \dots i_k})$  is a  $k$ th order  $n$ -dimensional symmetric, essentially nonpositive tensor, with  $k \geq 2$ . If  $R_{\min}(\mathcal{A}) \geq 0$ , for  $i = 1, \dots, n$ , then  $\mathcal{A}$  is copositive. If  $R_{\min}(\mathcal{A}) > 0$ , for  $i = 1, \dots, n$ , then  $\mathcal{A}$  is strictly copositive.

Finally, we may prove the following theorem.

**Theorem 10** Suppose that  $\mathcal{B} = (b_{i_1 \dots i_k})$  is a  $k$ th order  $n$ -dimensional real symmetric tensor, with  $k \geq 2$ . If

$$b_{i \dots i} + \sum \{b_{ii_2 \dots i_k} : b_{ii_2 \dots i_k} < 0, (i, i_2, \dots, i_k) \neq (i, \dots, i)\} \geq 0, \quad (11)$$

for  $i = 1, \dots, n$ , then  $\mathcal{B}$  is copositive. If

$$b_{i \dots i} + \sum \{b_{ii_2 \dots i_k} : b_{ii_2 \dots i_k} < 0, (i, i_2, \dots, i_k) \neq (i, \dots, i)\} > 0, \quad (12)$$

for  $i = 1, \dots, n$ , then  $\mathcal{B}$  is strictly copositive.

**Proof.** Construct a  $k$ th order  $n$ -dimensional real symmetric tensor  $\mathcal{A} = (a_{i_1 \dots i_k})$  by  $a_{i_1 \dots i_k} = 0$  if  $b_{i_1 \dots i_k} > 0$  and  $(i_1, \dots, i_k) \neq (i_1, \dots, i_1)$ , and  $a_{i_1 \dots i_k} = b_{i_1 \dots i_k}$  otherwise. Then  $\mathcal{A}$  is symmetric and essentially nonpositive, and  $\mathcal{A} \leq \mathcal{B}$ . Now the conclusions follow from Theorem 5 (c) and Lemma 9.  $\square$

We may call a real symmetric tensor satisfying (11) a **nonnegative diagonal dominated tensor**, and a real symmetric tensor satisfying (12) a **positive diagonal dominated tensor**. We see that a symmetric nonnegative tensor is a nonnegative diagonal dominated tensor. The Laplacian tensor of a uniform hypergraph, introduced in [15], is a nonnegative diagonal dominated tensor. Thus, the adjacency tensor, the Laplacian tensor and the signless Laplacian tensor of a uniform hypergraph are examples of copositive tensors.

## 6 Further Properties of Copositive Tensors

The proofs of many properties of copositive matrices may not be extended to copositive tensors directly. This leaves some puzzles: do such properties of copositive matrices still hold for copositive tensors? The situation is in particular odd when the order  $k$  is odd.

The first question is: Does a copositive tensor have an H-eigenvalue? When the order  $k$  is even, by [14], a real symmetric  $k$ th order  $n$ -dimensional tensor  $\mathcal{A}$  always has an H-eigenvalue. Thus, we may ask

**Question 1.** When the order  $k$  is odd, does a copositive tensor  $\mathcal{A}$  always have an H-eigenvalue?

**Question 2.** When the order  $k \geq 3$ , if a copositive tensor  $\mathcal{A}$  has at least one H-eigenvalue, does it always have a nonnegative H-eigenvalue?

For a copositive matrix  $A$ , Haynsworth and Hoffman [7] showed that its largest eigenvalue  $\lambda$  satisfies that  $\lambda \geq |\mu|$ , where  $\mu$  is any other eigenvalue of  $A$ . It is not clear if this is still true for copositive tensors.

**Question 3.** When the order  $k \geq 3$ , if a copositive tensor  $\mathcal{A}$  has a nonnegative H-eigenvalue  $\lambda$ , does it satisfy  $\lambda \geq |\mu|$ , where  $\mu$  is any other H-eigenvalue of  $\mathcal{A}$ ?

If  $\mathcal{A}$  is a symmetric nonnegative tensor, then the answers to the above three questions are all “yes”. If  $\mathcal{A}$  is a symmetric, essentially nonpositive tensor, and  $R_{\min}(\mathcal{A}) \geq 0$ , then the answers to the above three questions are also all “yes”.

When  $k$  is even, if all the H-eigenvalues of a real symmetric tensor are nonnegative, then that tensor is positive semi-definite, thus copositive. The situation becomes again puzzled when  $k$  is even.

**Question 4.** Suppose that the order  $k$  is odd and all the H-eigenvalues of a real symmetric tensor are nonnegative. Is that tensor copositive?

No matter such basic questions remain open, we may derive some further properties of a copositive tensor.

**Proposition 11** *If a copositive tensor  $\mathcal{A}$  has an  $H^+$ -eigenvalue  $\lambda$ , then  $\lambda \geq 0$ .*

**Proof.** Then we have  $\mathcal{A}x^{k-1} = \lambda x^{[k-1]}$  with  $x \in \mathbb{R}_+^n, x \neq 0$ . We have  $\lambda = \frac{\mathcal{A}x^k}{\sum_{i=1}^n x_i^k} \geq 0$ .  $\square$

We now extend one theorem of Väliäho, Theorem 3.2 of [16], to copositive tensors.

**Proposition 12** *Suppose that  $\mathcal{A}$  is a  $k$ th order  $n$ -dimensional copositive tensor. Then  $x \in \mathbb{R}_+^n$  and  $\mathcal{A}x^k = 0$  imply that  $\mathcal{A}x^{k-1} \geq 0$ .*

**Proof.** Consider  $f(x) = \mathcal{A}x^k$ . If  $\mathcal{A}x^k = 0$  for some  $x \in \mathbb{R}_+^n$ , then for  $t > 0$  and  $i = 1, \dots, n$ ,  $\mathcal{A}(x + te^{(i)})^k \geq 0$  as  $\mathcal{A}$  is copositive and  $x + te^{(i)} \in \mathbb{R}_+^n$ . This implies that  $f'(x) = k\mathcal{A}x^{k-1} \geq 0$ .  $\square$

However, it is not clear if the next theorem of Väliäho, Theorem 3.3 of [16], can be extended to copositive tensors or not. This leaves another puzzle.

It is easy to see that if  $\mathcal{A}$  and  $\mathcal{B}$  are two (strictly) copositive tensors of the same order and dimension, then  $\mathcal{A} + \mathcal{B}$  is also a (strictly) copositive tensor, and if  $\mathcal{A}$  is a (strictly) copositive tensor and  $\alpha$  is a positive number, then  $\alpha\mathcal{A}$  is also a (strictly) copositive tensor. Then all copositive tensors of order  $k$  and dimension  $n$  form a convex cone. We denote it by  $C_{k,n}$ . Similarly, all strictly copositive tensors of order  $k$  and dimension  $n$  form a convex cone. We denote it by  $SC_{k,n}$ . Similarly, we have the positive semi-definite tensor cone of order  $k$  and dimension  $n$ , denoted by  $PSD_{k,n}$ , and the nonnegative diagonal dominated tensor cone of order  $k$  and dimension  $n$ , denoted by  $NDD_{k,n}$ , etc. When  $k$  is odd,  $NDD_{k,n}$  is a subcone of  $C_{k,n}$ . When  $k$  is even,  $NDD_{k,n}$  and  $PSD_{k,n}$  are two subcones of  $C_{k,n}$ .

Even when  $k$  is odd, a copositive tensor may not be a nonnegative diagonal dominated tensor. For example, let  $k = n = 3$ ,  $a_{113} = a_{131} = a_{311} = a_{223} = a_{232} = a_{322} = 2$ ,  $a_{123} = a_{132} = a_{213} = a_{231} = a_{312} = a_{321} = -1$ , and the other elements of  $\mathcal{A}$  be zero. Then  $\mathcal{A}x^3 = 6(x_1^2 + x_2^2 - x_1x_2)x_3 \geq 0$  for any  $x \in \mathbb{R}_+^3$ , i.e.,  $\mathcal{A}$  is a copositive tensor. But  $\mathcal{A}$  is not a nonnegative diagonal dominated tensor, as all diagonal elements of  $\mathcal{A}$  are zero, but there are negative off-diagonal elements.

**Proposition 13**  *$SC_{k,n}$  is exactly the interior cone of  $C_{k,n}$ .*

**Proof.** Denote  $B_{k,n}$  as the set of all  $k$ th order  $n$ -dimensional real symmetric tensors whose Frobenius norms are 1. Suppose that  $\mathcal{A} \in SC_{k,n}$ . Let  $\mathcal{A}(t, \mathcal{B}) = \mathcal{A} + t\mathcal{B}$ , where  $\mathcal{B} \in B_{k,n}$ . Let  $\delta$  be a positive number,  $0 \leq t \leq \delta$ . Then by (9),

$$|N_{\min}(\mathcal{A}(t, \mathcal{B})) - N_{\min}(\mathcal{A})| \leq c\delta,$$

where  $c$  is a certain norm ratio constant. Thus, we have some  $\delta > 0$ , such that for all  $\mathcal{B} \in B_{k,n}$  and  $0 \leq t \leq \delta$ ,  $\mathcal{A}(t, \mathcal{B}) \in SC_{k,n}$ . This shows that  $SC_{k,n}$  is in the interior of  $C_{k,n}$ .

On the other hand, suppose that  $\mathcal{A} \in C_{k,n} \setminus SC_{k,n}$ . By Theorem 5,  $N_{\min}(\mathcal{A}) = 0$ . Then there is a  $y \in \mathbb{R}_+^n$ , such that  $\sum_{i=1}^n y_i^k = 1$  and  $\mathcal{A}y^k = 0$ . Let  $\mathcal{A}(t) = \mathcal{A} - ty^k$ . Then we see that  $N_{\min}(\mathcal{A}(t)) < 0$  for all  $t > 0$ . Thus,  $\mathcal{A}$  is not in the interior of  $C_{k,n}$ . This completes our proof.  $\square$

It is well-known that the copositive matrix cone and the completely positive matrix cone are dual to each other [1, 8, 17]. This was established by Hall and Newman [6]. We may consider this issue in the tensor case. Let  $y \in \mathbb{R}_+^n$ . Then we may regard  $y^k$  as a rank-one  $k$ th order  $n$ -dimensional completely positive tensor  $y^k = (y_{i_1} \cdots y_{i_k})$ . We call a  $k$ th order  $n$ -dimensional tensor  $\mathcal{A}$  a **completely positive tensor** if there are  $y^{(1)}, \dots, y^{(r)} \in \mathbb{R}_+^n$  such that

$$\mathcal{A} = \sum_{i=1}^r (y^{(i)})^k.$$

The smallest value of  $r$  to make the above expression hold is called the **CP-rank** of  $\mathcal{A}$ . Clearly, a completely positive tensor is a symmetric nonnegative tensor, and all the  $k$ th order  $n$ -dimensional completely positive tensors form a convex cone, the completely positive tensor cone, denoted as  $CP_{k,n}$ . For two  $k$ th order  $n$ -dimensional real symmetric tensors  $\mathcal{A} = (a_{i_1 \dots i_k})$  and  $\mathcal{B} = (b_{i_1 \dots i_k})$ , denote its inner product as

$$\langle \mathcal{A}, \mathcal{B} \rangle = \sum_{i_1, \dots, i_k=1}^n a_{i_1 \dots i_k} b_{i_1 \dots i_k}.$$

Denote the space of all  $k$ th order  $n$ -dimensional real symmetric tensors as  $S_{k,n}$ . For a convex cone  $K$  in  $S_{k,n}$ , its dual cone is defined as

$$K^* = \{\mathcal{B} \in S_{k,n} : \langle \mathcal{A}, \mathcal{B} \rangle \geq 0\}.$$

We have  $K^{**} = \text{cl}K$ . If  $K$  is closed, then we have  $K^{**} = K$ . By the definition of copositive tensors, we have  $C_{k,n} = CP_{k,n}^*$ . Then we have  $C_{k,n}^* = \text{cl}CP_{k,n}$ . If  $CP_{k,n}$  is closed, then we have  $C_{k,n}^* = CP_{k,n}$ . We leave this as a future research topic.

## 7 Final Remarks

In Sections 3 and 4, we established two new spectral properties of symmetric nonnegative tensors. This shows that there are still unexplored topics of the spectral theory of symmetric nonnegative tensors. In Section 5, we introduced copositive tensors and strictly copositive tensors. Symmetric nonnegative tensors and positive semi-definite tensors are copositive tensors. Beside some simple properties of copositive tensors and strictly copositive tensors, we show that nonnegative diagonal dominated tensors are copositive tensors, and positive diagonal dominated tensors are strictly copositive tensors. Section 6 shows that there are many puzzles unsolved for copositive tensors and strictly copositive tensors. Hence, this paper is only a starting point for studying copositive tensors and strictly copositive tensors.

## References

- [1] A. Berman and N. Shaked-Monderer, *Completely Positive Matrices*, World Scientific, Singapore, 2003.
- [2] K.C. Chang, K. Pearson and T. Zhang, “Perron Frobenius Theorem for nonnegative tensors”, *Commu. Math. Sci.*, 6 (2008) 507-520.
- [3] K.C. Chang, K. Pearson and T. Zhang, “Primitivity, the convergence of the NZQ method, and the largest eigenvalue for nonnegative tensors”, *SIAM J. Matrix Anal. Appl.*, 32 (2011) 806-819.

- [4] J. Cooper and A. Dutle, “Spectra of uniform hypergraphs”, *Lin. Alg. Appl.*, 436 (2012) 3268-3292.
- [5] S. Friedland, S. Gaubert and L. Han, “Perron-Frobenius theorem for nonnegative multilinear forms and extensions”, *Lin. Alg. Appl.*, 438 (2013), 738-749.
- [6] M. Hall and M. Newman, “Copositive and completely positive quadratic forms”, *Proc. Cambridge Philos. Soc.*, 59 (1963) 329-339.
- [7] E. Haynsworth and A.J. Hoffman, “Two remarks on copositive matrices”, *Lin. Alg. Appl.*, 2 (1969) 387-392.
- [8] J.-B. Hiriart-Urruty and A. Seeger, “A variational approach to copositive matrices”, *SIAM Review*, 52 (2010) 593-629.
- [9] S. Hu, Z. Huang and L. Qi, “Find the spectral radius of a nonnegative tensor”, arXiv:1111.2138v1 [math.NA] 9 Nov 2011.
- [10] L-H. Lim, *Singular values and eigenvalues of tensors: A variational approach*, Proceedings of the First IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing (CAMSAP), December 13-15, 129-132, 2005.
- [11] Y. Liu, G. Zhou and N.F. Ibrahim, “An always convergent algorithm for the largest eigenvalue of an irreducible nonnegative tensor”, *Journal of Computational and Applied Mathematics*, 235 (2010) 286-292.
- [12] T.S. Motzkin, “Copositive quadratic forms”, *National Bureau of Standards Report*, 1818 (1952) 11-12.
- [13] M. Ng, L. Qi and G. Zhou, “Finding the largest eigenvalue of a non-negative tensor”, *SIAM J. Matrix Anal. Appl.*, 31 (2009) 1090-1099.
- [14] L. Qi, “Eigenvalues of a real supersymmetric tensor”, *Journal of Symbolic Computation*, 40 (2005) 1302-1324.
- [15] L. Qi, “ $H^+$ -eigenvalues of Laplacian and signless Laplacian tensors”, Preprint, Department of Applied Mathematics, The Hong Kong Polytechnic University, November, 2012.
- [16] H. Väliäho, “Criteria of copositive matrices”, *Lin. Alg. Appl.*, 81 (1986) 19-34.
- [17] C. Xu, “completely positive matrices”, *Lin. Alg. Appl.*, 379 (2004), 319-327.

- [18] Q. Yang and Y. Yang, “Further results for Perron-Frobenius Theorem for non-negative tensors II”, *SIAM Journal on Matrix Analysis and Applications* **32** (2011) 1236-1250.
- [19] Y. Yang and Q. Yang, “Further results for Perron-Frobenius Theorem for non-negative tensors”, *SIAM J. Matrix Anal. Appl.*, 31 (2010) 2517-2530.
- [20] L. Zhang and L. Qi, “Linear convergence of an algorithm for computing the largest eigenvalue of a nonnegative tensor”, *Numerical Linear Algebra with Applications*, 19 (2012) 830-841.
- [21] L. Zhang, L. Qi and Z. Luo, The dominant eigenvalue of an essentially nonnegative tensor, arXiv: 1110.6261v1 [math.NA] 28 Oct 2011.
- [22] L. Zhang, L. Qi and Y. Xu, “Linear convergence of the LZI algorithm for weakly positive tensors”, *Journal of Computational Mathematics*, 30 (2012) 24-33.
- [23] T. Zhang, Private communication, September 2012.